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Axisymmetric thermal-creep flow of a slightly rarefied gas induced around two unequal spheres

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A thermal-creep flow of a slightly rarefied gas induced axisymmetrically around two unequal spheres is studied on the basis of kinetic theory. The spheres, whose thermal conductivities are assumed to be identical with that of the gas, for simplicity, are placed in an infinite expanse of the gas at rest with a uniform temperature gradient at a far distance. Owing to the poor thermal conductivities of the spheres, a tangential temperature gradient is established on the surfaces, and this causes a thermal-creep flow in its direction. Consequently, the spheres experience forces in the opposite direction.

The flow considered here is a low-Reynolds-number flow in the ordinary fluid-dynamic sense (except for the Knudsen layer), and the solution is obtained in terms of bispherical coordinates, with respect to which the system of equations of Stokes type is well developed. The velocity field around the spheres and the forces acting on them are given explicitly. The results show the interesting feature that the smaller sphere experiences the minimum force at a value of the separation distance that depends on the radius ratio. This is in contrast with the case of the axisymmetric motion of two spheres treated by Stimson & Jeffery (1926) in ordinary fluid dynamics at low Reynolds number.

The ultimate velocities that the spheres would have under the action of the present thermal force when they are freely suspended are also obtained by utilizing the results for the forces of axisymmetric translational problems of two spheres at low Reynolds number. For a given temperature gradient in the gas, both spheres acquire larger velocities than those they would have if they were alone, and the smaller sphere tends to move faster than the larger one in the direction opposite to the temperature gradient.

Also presented, for completeness, are the results for the sphere–plane case and for the case of eccentric spheres, the solutions for which are derived as special cases of the preceding problem of two unequal spheres.

1. Introduction

Fine particles of micron and submicron size, when dispersed in a gas with non-uniform temperature, often experience forces other than gravity or electrostatic force, owing to the rarefaction of the gas, when the mean free path of gas molecules (~ 0.1 μm in the standard state) is not negligible compared with the particle size. These forces, which are called thermal forces, are extensively utilized in many engineering devices such as precipitators for removing minute particles from gas or gas streams. These thermal precipitators are often said to be more effective than electrostatic ones (e.g. Talbot *et al.* 1980). As in the sedimentation problems treated by Batchelor (1972, 1982), when the dispersions are not extremely dilute, fluid-dynamic interactions between particles (mainly pairs of particles) become appreciable, thus affecting the thermal forces, and eventually the average movement of particles being removed. It therefore becomes necessary to study the fundamental problem of two-sphere interactions in a slightly rarefied gas flow: this is known as thermal creep. The present study is concerned with a part of this basic problem.

Now consider a body placed in a slightly rarefied gas. When the temperature of the body surface is not uniform, a flow will be induced around it in the direction of the temperature gradient of the surface owing to the rarefaction of the gas. If the body is freely suspended, it moves in the direction opposite to the temperature gradient. Such a flow is called a thermal-creep flow (Kennard 1938, p. 327), and for slightly rarefied gases this is the most important flow in addition to the one due to simple velocity slip on the surface of the body. Kennard first attempted an analysis for the steady thermal-creep flow over a plane wall from the viewpoint of kinetic theory, and later Sone (1966) and Kogan (1969, chap. V) gave rigorous analyses of the problem based on the Boltzmann–Krook–Welander equation (Bhatnager, Gross & Krook 1954; Welander 1954; Kogan 1969, pp. 79–83) together with a boundary condition of diffusive type. Studies have also been made under a more general boundary condition of specular–diffusive or Maxwell type (Onishi 1972*a*, 1973; Loyalka, Petrellis & Storvick 1975; for details of the boundary condition see e.g. Cercignani 1975, p. 119).

Onishi (1972*b*) also studied the thermal-creep flow induced around a single sphere when a uniform temperature gradient is imposed on the gas at infinity, and obtained explicitly the velocity field together with the force acting on the sphere. There, the thermal conductivity of the sphere was assumed to be of the order of that of the ambient gas. Owing to this relatively poor thermal conductivity of the sphere, a non-uniform temperature distribution was formed on the surface, and a flow was induced over it in the direction of the temperature gradient. Originally this problem was considered by Epstein (1929), who treated it as a problem of thermophoresis (see Waldmann 1961; Dwyer 1967; Gorelov 1976; Sone & Aoki 1977). His analysis was based on the so-called classical slip-flow theory, and therefore lacks rigour but gives qualitatively correct results. Talbot *et al.* (1980) review the theoretical and experimental work on the single-sphere problem, which is not necessarily restricted to small Knudsen numbers.

Here we shall extend the analysis for the single-sphere problem (Onishi 1972*b*) to a case in which two unequal-sized spheres are present in the gas, on which is imposed a uniform temperature gradient at infinity. We will obtain the velocity field around the spheres, the forces acting on them, and the velocities acquired by the spheres when they are free. This problem is a superposition of two basic problems, namely the axisymmetric problem where the line of centres of the spheres is parallel to the temperature gradient, and the asymmetric problem where the line of centres is at right angles to it. We consider only the former in the present paper; the latter will be discussed elsewhere. The analysis of the present problem will be carried out under the same assumptions as those for the single-sphere case; these are: (i) gas motion is described by the Boltzmann–Krook–Welander equation; (ii) the interaction of gas molecules with the surface of the body is of a Maxwellian distribution characterized by its temperature and velocity (diffuse boundary condition); (iii) the mean free path of gas molecules is small compared with the characteristic length of the body (i.e. the Knudsen number $\ll 1$); (iv) the deviation of the system from a stationary equilibrium reference state is small, and the problem can be linearized (this implies, together with assumption (iii), that the Reynolds number of the bulk flow field is negligibly small (see e.g. Sone 1971)); and (v) the heat flow inside the body obeys Fourier's law of heat conduction. In addition to assumptions (i)–(v), we further assume that the thermal conductivities of the spheres are identical with that of the ambient gas. This last assumption reduces the amount of troublesome calculations needed in dealing with the temperature field, while still enabling us to retain the qualitatively significant nature of the problem. Incidentally, we note that this assumption is not so unrealistic as it would seem. Actually, for some particles such as sawdust, paper, glasswool, silk, charcoal, sponge etc., which are commonly seen in air, their thermal conductivities are quite low and their ratio to those of common gases (e.g. air, N_2, Ar, He) is of order unity.

We have also considered the thermal-creep flow induced by a sphere in the presence of a plane wall, and the flow induced in a region bounded by two eccentric spheres, obtaining the velocity acquired by the free sphere as well as the velocity fields of the gas in both cases. These problems together with the one for two unequal spheres complete the axisymmetric-flow problem for thermal creep.

Although there still remain asymmetric-flow problems to be discussed in order to gain a comprehensive understanding of the interaction problem of thermal creep, the present work will serve to elucidate some of the fundamental aspects of this problem (see §§5 and 6).

2. Fundamental equations and the boundary conditions

The description of the general behaviour of gas around solid bodies has been given by Sone (1969, 1971) on the basis of assumptions (i)–(iv) stated in §1. The present analysis will be based upon this general theory. According to it, fluid-dynamic quantities such as velocity, pressure and temperature are expressed by the sum of two parts, the Hilbert part (with suffix H) and the Knudsen-layer part (with suffix K), each of which is obtained in an expansion in terms of the Knudsen number of the system (see also Sone & Aoki 1977; Sone & Onishi 1978; Onishi 1982):†

$$f = f_{\mathrm{H}} + f_{\mathrm{K}}, \tag{2.1}‡$$

$$f_{\mathrm{H}} = f_{\mathrm{H}}^{(0)} + kf_{\mathrm{H}}^{(1)} + k^2 f_{\mathrm{H}}^{(2)} + \dots, \quad f_{\mathrm{K}} = kf_{\mathrm{K}}^{(1)} + k^2 f_{\mathrm{K}}^{(2)} + \dots, \tag{2.2a, b}$$

where

$$k = \tfrac{1}{2}\pi^{\frac{1}{2}}\mathbb{K}. \tag{2.3}$$

f stands for any one of the perturbations of fluid-dynamic quantities from a stationary equilibrium reference state. $\mathbb{K}$ is the Knudsen number defined by $\mathbb{K} = l/L$, l being the mean free path of gas molecules at the reference state, and L a reference length of the system. f_{H} has a lengthscale of variation of the order of the characteristic length ($\sim L$), and f_{K} has a lengthscale of variation of the order of the mean free path of gas molecules. f_{K} is only appreciable within the Knudsen layer formed near the surface of the body, and therefore takes account of the correction to f_{H} there.

† In particular, for the details of the theory and the refined results, reference may be made to Sone & Onishi (1978), in which the general behaviour of a condensable gas is derived systematically from the Boltzmann–Krook–Welander equation.

‡ The decomposition of the solution into two parts for small k is a useful technique (Grad 1969). The accuracy of each part should be good enough when k is sufficiently small for terms involving $\exp(-1/k)$ in the solution to be negligible compared with any powers of k.

The Hilbert parts of the pressure, velocity, temperature and density satisfy a system of equations of Stokes type:

$$\nabla p_{\mathrm{H}}^{(0)} = 0, \quad \nabla \cdot \boldsymbol{u}_{\mathrm{H}}^{(m)} = 0, \tag{2.4), (2.5}$$

$$\nabla p_{\mathrm{H}}^{(m+1)} - \nabla^2 \boldsymbol{u}_{\mathrm{H}}^{(m)} = 0, \quad \nabla^2 \tau_{\mathrm{H}}^{(m)} = 0, \tag{2.6), (2.7}$$

$$\rho_{\mathrm{H}}^{(m)} = p_{\mathrm{H}}^{(m)} - \tau_{\mathrm{H}}^{(m)}, \tag{2.8}$$

where $m = 0, 1, 2, \ldots$. $P_0(1+p)$, $(2RT_0)^{\frac{1}{2}}\boldsymbol{u}$, $T_0(1+\tau)$ and $P_0(RT_0)^{-1}(1+\rho)$ are respectively the gas pressure, velocity vector, temperature and density. P_0 is a reference pressure, T_0 a reference temperature and R the gas constant per unit mass. $L^{-1}\nabla$ is the gradient operator for a rectangular coordinate system $L\boldsymbol{x}$. The viscosity μ of the gas at this reference state is related to the Knudsen number $\mathbb{K}$ as $\mu = P_0(8RT_0/\pi)^{-\frac{1}{2}} L\mathbb{K}$ (Vincenti & Kruger 1965, chap. X).

The boundary conditions on the surface of the body appropriate for the system of equations (2.5)–(2.7) are

$$\boldsymbol{u}_{\mathrm{H}}^{(0)} = \boldsymbol{u}_{\mathrm{W}}^{(0)}, \quad \tau_{\mathrm{H}}^{(0)} = \tau_{\mathrm{W}}^{(0)}, \tag{2.9), (2.10}$$

$$(\boldsymbol{u}_{\mathrm{H}}^{(1)} - \boldsymbol{u}_{\mathrm{W}}^{(1)}) \cdot \boldsymbol{t} = -k_0 \boldsymbol{n} \cdot [\nabla \boldsymbol{u}_{\mathrm{H}}^{(0)} + (\nabla \boldsymbol{u}_{\mathrm{H}}^{(0)})^T] \cdot \boldsymbol{t} - K_1 \boldsymbol{t} \cdot \nabla \tau_{\mathrm{H}}^{(0)}, \tag{2.11a}$$

$$\boldsymbol{u}_{\mathrm{H}}^{(1)} \cdot \boldsymbol{n} = 0, \tag{2.11b}$$

$$\tau_{\mathrm{H}}^{(1)} - \tau_{\mathrm{W}}^{(1)} = d_1 \boldsymbol{n} \cdot \nabla \tau_{\mathrm{H}}^{(0)}, \tag{2.12}$$

where $k_0 = -1.016191$, $K_1 = -0.383161$ and $d_1 = 1.302716$ (these numerical values are taken from Sone & Onishi 1978). $\boldsymbol{n}$ and $\boldsymbol{t}$ are respectively the outward unit normal vector (pointed into the gas) and a unit tangential vector to the surface. $(2RT_0)^{\frac{1}{2}}\boldsymbol{u}_{\mathrm{W}}$ and $T_0(1+\tau_{\mathrm{W}})$ denote the surface velocity (with $\boldsymbol{u}_{\mathrm{W}} \cdot \boldsymbol{n} = 0$) and surface temperature of the body, $\boldsymbol{u}_{\mathrm{W}}$ and τ_{W} being expanded as in (2.2a). $(\nabla\boldsymbol{u})^{\mathrm{T}}$ denotes the transpose of $\nabla\boldsymbol{u}$. The second term on the right-hand side of (2.11a) is the driving force for the thermal-creep flow.

The Knudsen-layer parts of the velocity, density and temperature are, near the surface of the body,

$$\boldsymbol{u}_{\mathrm{K}}^{(1)} \cdot \boldsymbol{t} = -Y_0 \boldsymbol{n} \cdot [\nabla \boldsymbol{u}_{\mathrm{H}}^{(0)} + (\nabla \boldsymbol{u}_{\mathrm{H}}^{(0)})^T] \cdot \boldsymbol{t} - \tfrac{1}{2} Y_1 \boldsymbol{t} \cdot \nabla \tau_{\mathrm{H}}^{(0)}, \quad \boldsymbol{u}_{\mathrm{K}}^{(1)} \cdot \boldsymbol{n} = 0, \tag{2.13a, b}$$

$$\rho_{\mathrm{K}}^{(1)} = \Omega_1 \boldsymbol{n} \cdot \nabla \tau_{\mathrm{H}}^{(0)}, \quad \tau_{\mathrm{K}}^{(1)} = \Theta_1 \boldsymbol{n} \cdot \nabla \tau_{\mathrm{H}}^{(0)}. \tag{2.14), (2.15}$$

Here Y_0, Y_1, Ω_1 and Θ_1 are the universal functions of η, a stretched coordinate along $\boldsymbol{n}$, defined as $(\boldsymbol{x} - \boldsymbol{x}_{\mathrm{W}})/k = \boldsymbol{n}\eta$, where $\boldsymbol{x}_{\mathrm{W}}$ represents the surface of the body (for the numerical values of these functions see Sone & Onishi 1978).

Next, inside the body, when assumption (v) of §1 holds, we have the Laplace equation for the temperature field:

$$\nabla^2 \tilde{\tau} = 0, \tag{2.16}$$

where $T_0(1+\tilde{\tau})$ denotes the temperature of the body and $\tilde{\tau}$ has an expansion of the form (2.2a).

To summarize, we have only to solve (2.4)–(2.7) and (2.16) under the following conditions;

(i) conditions on the surface of the body, (2.9)–(2.12);

(ii) conditions at infinity for p_{H}, $\boldsymbol{u}_{\mathrm{H}}$ and τ_{H};

(iii) continuity condition for heat flow on the surface of the body, i.e.

$$-\boldsymbol{n} \cdot \nabla \tilde{\tau} = \frac{4}{5} \frac{\lambda_{\mathrm{g}}}{\lambda_{\mathrm{s}}} \boldsymbol{Q} \cdot \boldsymbol{n} \frac{1}{k}, \tag{2.17}$$

where λ_s and λ_g are the thermal conductivities of the body and the gas respectively, λ_g being related to $\mathbb{K}$ as $\lambda_g = \frac{5}{2}RP_0(8RT_0/\pi)^{-\frac{1}{2}}L\mathbb{K}$ (Vincenti & Kruger 1965, chap. X). Here we have introduced a factor λ_g/λ_s in (2.17) for generality, but it is noted that we have assumed $\lambda_s = \lambda_g$ for simplicity in the present problem. $P_0(2RT_0)^{\frac{1}{2}}\boldsymbol{Q}$ is the heat-flux vector of gas, and is given by

$$\begin{aligned} \boldsymbol{Q}\cdot\boldsymbol{n} &= k(\boldsymbol{Q}_{\mathrm{H}}^{(1)}+\boldsymbol{Q}_{\mathrm{K}}^{(1)})\cdot\boldsymbol{n}+k^2(\boldsymbol{Q}_{\mathrm{H}}^{(2)}+\boldsymbol{Q}_{\mathrm{K}}^{(2)})\cdot\boldsymbol{n}+\dots \\ &= -\tfrac{5}{4}\boldsymbol{n}\cdot\boldsymbol{\nabla}\tau_{\mathrm{H}}^{(0)}k+O(k^2). \end{aligned} \tag{2.18}$$

The term $O(k^2)$ is not written down explicitly, because it does not affect the velocity field up to the order we are concerned with (i.e. to order k).

3. The problem and the coordinate system

Two spheres with radius a_1 (labelled S_1) and radius a_2 (labelled S_2) are placed in an infinite expanse of a slightly rarefied gas. The gas is at rest at infinity, and has there a uniform pressure and a uniform temperature gradient parallel to the line of centres of the spheres. We introduce the cylindrical coordinate system (Lr, ϕ, Lz) with the z-axis being taken to be coincident with this line of centres, and also the bispherical coordinate system (ξ, ϕ, α), i.e.

$$r = \frac{c\sin\alpha}{\cosh\xi-\cos\alpha}, \quad z = \frac{c\sinh\xi}{\cosh\xi-\cos\alpha}, \quad (0 \leqq \alpha \leqq \pi, \ -\infty < \xi < \infty). \tag{3.1}$$

ξ = constant denotes a spherical surface with its centre on the z-axis, and, according to whether the constant is positive or negative, the centre lies on the positive or negative side of the z-axis. $\xi = 0$ represents the plane at $z = 0$, and $\xi \to \pm\infty$ the point at $z = \pm c$. α = constant is the arc of a circle, terminating at $z = \pm c$ and having its centre in the plane $z = 0$. $\alpha = 0$ is the segment of the z-axis for which $|z| \geqq c$, and $\alpha = \pi$ the segment for which $|z| \leqq c$. c is the scale factor related to the radius of the sphere and also to the distance between the centre of the sphere and the origin of the coordinate system, namely

$$\left.\begin{aligned} a_1 &= \frac{Lc}{\sinh\xi_1}, \quad Z_1 = Lc\coth\xi_1, \\ a_2 &= \frac{Lc}{\sinh\xi_2}, \quad Z_2 = Lc\coth\xi_2, \end{aligned}\right\} \tag{3.2}$$

where ξ_1 and ξ_2 are positive constants, with $\xi = \xi_1$ denoting S_1 and $\xi = -\xi_2$ denoting S_2. If $\xi_2 < \xi_1$, and S_2 is described by $\xi = \xi_2$, then S_2 encloses within it S_1. Z_1 and Z_2 are the distances between the origin of the coordinate system and the centre of each sphere (figure 1).

Now we define a reference state for the present analysis. We choose a_1, the radius of S_1, as L, and the pressure at infinity as P_0. Also we adopt as T_0 the temperature of the gas at $z = 0$ and $r \to \infty$. With this reference state, we can write down conditions (ii) and (iii) of §2:

$$\boldsymbol{u}_{\mathrm{H}} \to 0, \quad p_{\mathrm{H}} \to 0, \quad \tau_{\mathrm{H}} \to Bz \quad \text{at infinity}, \tag{3.3}$$

and on both S_1 and S_2

$$\boldsymbol{n}\cdot\boldsymbol{\nabla}\tilde{\tau}^{(0)} = \boldsymbol{n}\cdot\boldsymbol{\nabla}\tau_{\mathrm{H}}^{(0)}, \quad \boldsymbol{n}\cdot\boldsymbol{\nabla}\tilde{\tau}^{(1)} = \boldsymbol{n}\cdot\boldsymbol{\nabla}\tau_{\mathrm{H}}^{(1)}+\dots, \tag{3.4a, b}$$

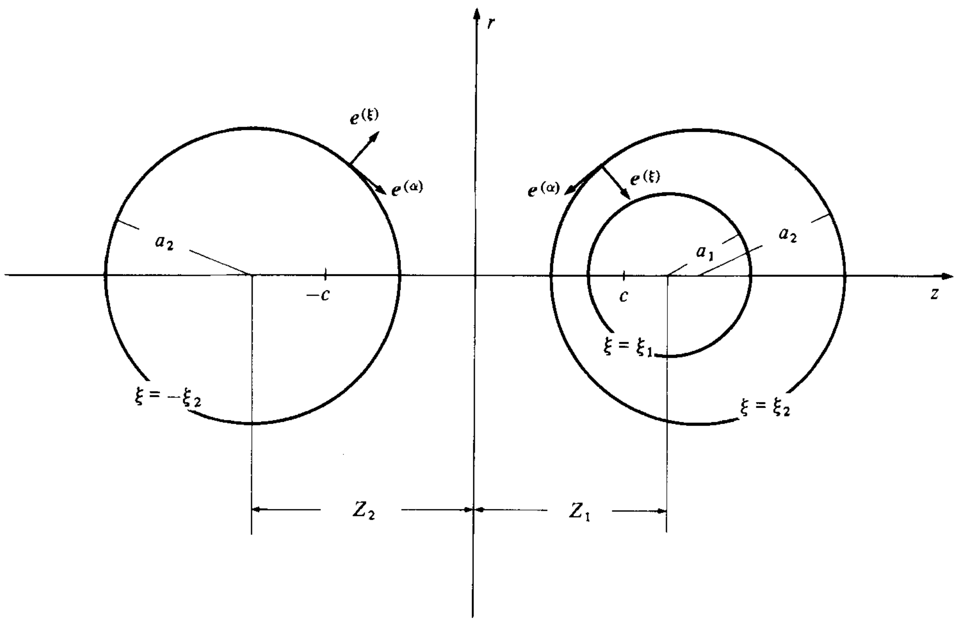

FIGURE 1. Schematic view of the problem and the bispherical coordinate system (ξ, ϕ, α).

where the assumption $\lambda_s = \lambda_g$ is made, and $B = a_1 T_0^{-1}(\partial T/\partial Z)_\infty$, $(\partial T/\partial Z)_\infty$ being the dimensional uniform temperature gradient at infinity. It should be noted again that (3.4*b*) does not affect the velocity field up to $O(k)$, and therefore its complete expression is omitted.

4. Analysis

The solutions to (2.4)–(2.7) and (2.16) with $m = 0$ that satisfy (2.9), (2.10), (3.3) and (3.4*a*) are easy to obtain. They are

$$p_{\mathrm{H}}^{(0)} = 0, \quad \boldsymbol{u}_{\mathrm{H}}^{(0)} = 0, \quad p_{\mathrm{H}}^{(1)} = 0, \quad \tau_{\mathrm{H}}^{(0)} = Bz, \quad \tilde{\tau}^{(0)} = Bz. \tag{4.1}$$

We proceed with the next-order solutions (with $m = 1$). Introducing a stream function ψ which satisfies

$$\boldsymbol{u}_{\mathrm{H}}^{(1)} \cdot \boldsymbol{e}^{(\xi)} = -\frac{h}{r}\frac{\partial \psi}{\partial \alpha}, \quad \boldsymbol{u}_{\mathrm{H}}^{(1)} \cdot \boldsymbol{e}^{(\alpha)} = \frac{h}{r}\frac{\partial \psi}{\partial \xi}, \tag{4.2}$$

and eliminating $p_{\mathrm{H}}^{(2)}$ in (2.6), we have

$$\mathscr{L}^4(\psi) = 0 \tag{4.3}$$

in place of (2.6), where

$$\left.\begin{aligned} \mathscr{L}^2 &\equiv h\left[\frac{\partial}{\partial \xi}\left(h\frac{\partial}{\partial \xi}\right) + (1-\beta^2)\frac{\partial}{\partial \beta}\left(h\frac{\partial}{\partial \beta}\right)\right], \\ h &= \frac{\cosh \xi - \beta}{c}, \quad \beta = \cos\alpha, \end{aligned}\right\} \tag{4.4}$$

and $\boldsymbol{e}^{(\xi)}$ and $\boldsymbol{e}^{(\alpha)}$ denote the unit vectors on the surface ξ = constant in the directions of increasing ξ and α respectively. They are orthogonal to each other, and

$$\left.\begin{aligned} \boldsymbol{e}^{(\xi)} &= -\boldsymbol{n}, \quad \boldsymbol{e}^{(\alpha)} = -\boldsymbol{t} \quad \text{on} \quad \xi = \xi_1, \\ \boldsymbol{e}^{(\xi)} &= \boldsymbol{n}, \quad \boldsymbol{e}^{(\alpha)} = \boldsymbol{t} \quad \text{on} \quad \xi = -\xi_2. \end{aligned}\right\} \tag{4.5}$$

The general solution to (4.3) is known to be (Stimson & Jeffery 1926)

$$\psi = (\cosh\xi-\beta)^{-\frac{3}{2}} \sum_{n=0}^{\infty} [A_n \cosh(n-\tfrac{1}{2})\xi + B_n \sinh(n-\tfrac{1}{2})\xi + C_n \cosh(n+\tfrac{3}{2})\xi + D_n \sinh(n+\tfrac{3}{2})\xi]\, V_n(\beta), \quad (4.6)$$

where $V_n(\beta) = P_{n-1}(\beta) - P_{n+1}(\beta)$, $P_n(\beta)$ being the Legendre polynomial of order n. A_n, B_n, C_n and D_n are all constants to be determined by the conditions for ψ, which can be readily written down from (2.11) with (4.1) being taken into account, i.e.

$$\psi = 0, \quad (4.7)$$

$$\frac{\partial\psi}{\partial\xi} = -K_1 Br\frac{\partial z}{\partial\alpha},$$

$$= K_1 Bc^2 \sinh\xi \frac{1-\beta^2}{(\cosh\xi-\beta)^3} \quad (4.8)$$

on S_1 and S_2. Expanding $(1-\beta^2)(\cosh\xi-\beta)^{-\frac{3}{2}}$ in terms of $V_n(\beta)$, we obtain the expansion form of (4.8) as

$$\frac{\partial\psi}{\partial\xi} = (\cosh\xi-\beta)^{-\frac{3}{2}} \sum_{n=1}^{\infty} \tfrac{1}{2} U_n(\xi)\, V_n(\beta), \quad (4.9a)$$

$$U_n(\xi) = 4\sqrt{2}\, K_1 Bc^2 \sinh\xi \frac{n(n+1)}{2n+1} \mathrm{e}^{-(n+\frac{1}{2})|\xi|}, \quad (4.9b)$$

which is in accordance with (4.6).

Applying (4.7) and (4.9) to (4.6), we have

$$\left.\begin{aligned} A_n &= (A_{31}b_3 + A_{41}b_4)/\Delta_n, \quad B_n = (A_{32}b_3 + A_{42}b_4)/\Delta_n,\\ C_n &= (A_{33}b_3 + A_{43}b_4)/\Delta_n, \quad D_n = (A_{34}b_3 + A_{44}b_4)/\Delta_n, \end{aligned}\right\} \quad (4.10)$$

where $b_3 = U_n(\xi_1)$, $b_4 = U_n(\xi_2)$, and

$$\left.\begin{aligned}
\Delta_n &= 16\sinh^2(n+\tfrac{1}{2})(\xi_1+\xi_2) - 4(2n+1)^2\sinh^2(\xi_1+\xi_2),\\
A_{31} &= (2n-1)\cosh(n-\tfrac{1}{2})\xi_2 \sinh(n+\tfrac{3}{2})(\xi_1+\xi_2)\\
&\quad -(2n+3)[\sinh(n-\tfrac{1}{2})\xi_1 + \sinh(n-\tfrac{1}{2})\xi_2\cosh(n+\tfrac{3}{2})(\xi_1+\xi_2)],\\
A_{41} &= (2n-1)\cosh(n-\tfrac{1}{2})\xi_1 \sinh(n+\tfrac{3}{2})(\xi_1+\xi_2)\\
&\quad -(2n+3)[\sinh(n-\tfrac{1}{2})\xi_2 + \sinh(n-\tfrac{1}{2})\xi_1\cosh(n+\tfrac{3}{2})(\xi_1+\xi_2)],\\
A_{32} &= (2n-1)\sinh(n-\tfrac{1}{2})\xi_2 \sinh(n+\tfrac{3}{2})(\xi_1+\xi_2)\\
&\quad +(2n+3)[\cosh(n-\tfrac{1}{2})\xi_1 - \cosh(n-\tfrac{1}{2})\xi_2\cosh(n+\tfrac{3}{2})(\xi_1+\xi_2)],\\
A_{42} &= -(2n-1)\sinh(n-\tfrac{1}{2})\xi_1 \sinh(n+\tfrac{3}{2})(\xi_1+\xi_2)\\
&\quad -(2n+3)[\cosh(n-\tfrac{1}{2})\xi_2 - \cosh(n-\tfrac{1}{2})\xi_1\cosh(n+\tfrac{3}{2})(\xi_1+\xi_2)],\\
A_{33} &= (2n+3)\cosh(n+\tfrac{3}{2})\xi_2 \sinh(n-\tfrac{1}{2})(\xi_1+\xi_2)\\
&\quad -(2n-1)[\sinh(n+\tfrac{3}{2})\xi_1 + \sinh(n+\tfrac{3}{2})\xi_2\cosh(n-\tfrac{1}{2})(\xi_1+\xi_2)],\\
A_{43} &= (2n+3)\cosh(n+\tfrac{3}{2})\xi_1 \sinh(n-\tfrac{1}{2})(\xi_1+\xi_2)\\
&\quad -(2n-1)[\sinh(n+\tfrac{3}{2})\xi_2 + \sinh(n+\tfrac{3}{2})\xi_1\cosh(n-\tfrac{1}{2})(\xi_1+\xi_2)],\\
A_{34} &= (2n+3)\sinh(n+\tfrac{3}{2})\xi_2 \sinh(n-\tfrac{1}{2})(\xi_1+\xi_2)\\
&\quad +(2n-1)[\cosh(n+\tfrac{3}{2})\xi_1 - \cosh(n+\tfrac{3}{2})\xi_2\cosh(n-\tfrac{1}{2})(\xi_1+\xi_2)],\\
A_{44} &= -(2n+3)\sinh(n+\tfrac{3}{2})\xi_1 \sinh(n-\tfrac{1}{2})(\xi_1+\xi_2)\\
&\quad -(2n-1)[\cosh(n+\tfrac{3}{2})\xi_2 - \cosh(n+\tfrac{3}{2})\xi_1\cosh(n-\tfrac{1}{2})(\xi_1+\xi_2)].
\end{aligned}\right\} \quad (4.11)$$

Δ_n and the A_{ij} are respectively the determinant and the cofactors of the matrix whose inverse multiplied by the column vector with components $(0, 0, b_3, b_4)$ yields the solution vector (A_n, B_n, C_n, D_n).

When the two spheres are equal, i.e. $\xi_2 = \xi_1$, the results become simple, giving

$$\left.\begin{aligned} A_n &= -4\sqrt{2}\,K_1 Bc^2 \frac{n(n+1)}{2n+1}\,\frac{\sinh\xi_1 \cosh(n+\frac{3}{2})\,\xi_1\, e^{-(n+\frac{1}{2})\xi_1}}{2\sinh(2n+1)\,\xi_1 + (2n+1)\sinh 2\xi_1},\\ C_n &= -\frac{\cosh(n-\frac{1}{2})\,\xi_1}{\cosh(n+\frac{3}{2})\,\xi_1} A_n,\\ B_n &= D_n = 0. \end{aligned}\right\} \tag{4.12}$$

The total velocity field (the Knudsen-layer correction included) now becomes

$$V_r = H_0\left[-\frac{h}{r}\frac{\partial\psi}{\partial\alpha}\right] + H_1\left[\frac{h}{r}\frac{\partial\psi}{\partial\xi} - \frac{B}{2}\left(h\frac{\partial z}{\partial\alpha}\right)_S Y_1(\eta)\right], \tag{4.13a}$$

$$V_z = H_1\left[\frac{h}{r}\frac{\partial\psi}{\partial\alpha}\right] + H_0\left[\frac{h}{r}\frac{\partial\psi}{\partial\xi} - \frac{B}{2}\left(h\frac{\partial z}{\partial\alpha}\right)_S Y_1(\eta)\right], \tag{4.13b}$$

$$H_0 = -\frac{\sinh\xi\sin\alpha}{ch}, \quad H_1 = \frac{\cosh\xi\cos\alpha - 1}{ch}, \quad (H_0^2 + H_1^2 = 1),$$

where V_r and V_z are the components of the non-dimensional gas velocity (non-dimensionalized with respect to $(2RT_0)^{\frac{1}{2}}$) in the directions of r and z respectively. $h(\partial z/\partial\alpha) = -(r\sinh\xi)/c$, and $(\ )_S$ denotes the value evaluated on the sphere surfaces. Hence

$$\left(h\frac{\partial z}{\partial\alpha}\right)_S = -r = -[1-(z-z_1)^2]^{\frac{1}{2}} \quad \text{on} \quad S_1\,(\xi = \xi_1), \tag{4.14a}$$

$$= \frac{r}{\lambda} = \frac{1}{\lambda}[\lambda^2 - (z+z_2)^2]^{\frac{1}{2}} \quad \text{on} \quad S_2\,(\xi = -\xi_2), \tag{4.14b}$$

where $\lambda = a_2/a_1$, and we have put $z_1 = Z_1/a_1$ and $z_2 = Z_2/a_1$. The stretched coordinate η is given in this case as follows:

$$\eta = \{[r^2 + (z-z_1)^2]^{\frac{1}{2}} - 1\}/k \quad \text{near} \quad S_1\,(\xi = \xi_1), \tag{4.15a}$$

$$\eta = \{[r^2 + (z+z_2)^2]^{\frac{1}{2}} - \lambda\}/k \quad \text{near} \quad S_2\,(\xi = -\xi_2). \tag{4.15b}$$

Some of the numerical values of $Y_1(\eta)$ are $Y_1(0) = 0.54777$, $Y_1(0.1) = 0.43961$, $Y_1(1.0) = 0.16720$, $Y_1(5.0) = 0.01256$ (see Sone & Onishi 1978).

We show some of the streamline patterns around two spheres in figures 2 and 3, where G denotes the minimum gap between the spheres. These patterns are based on the Hilbert part of the velocity (or ψ) only, and therefore need the Knudsen-layer corrections near the surfaces of the spheres. It would be sufficient, however, to get an overall idea of the flow around the spheres induced by the temperature gradient on the surfaces.

Once these solutions are obtained, we can consider two other cases which are quite interesting.

4.1. *Sphere–plane case* (*when S_2 reduces to a plane wall*)

By letting $\xi_2 \to 0$ we can get S_2 to reduce to the plane at $z = 0$. By this limit process the tangential temperature gradient disappears on the plane and a uniform

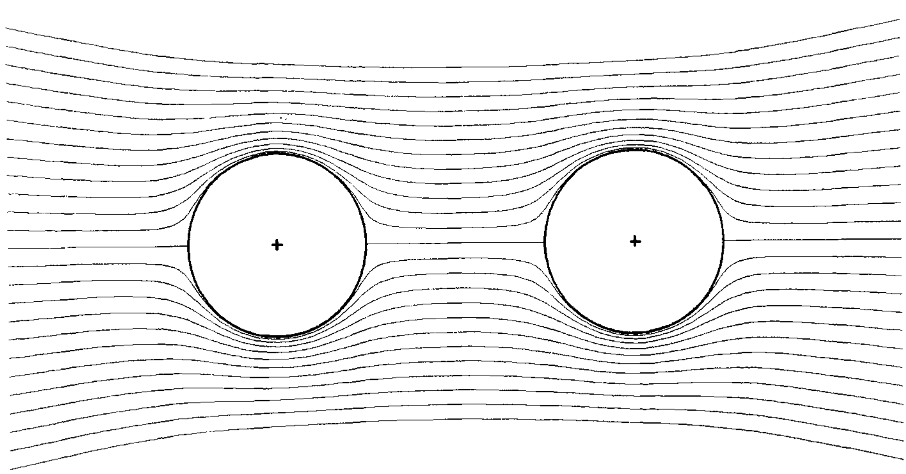

FIGURE 2. The streamlines for thermal-creep flow around two spheres ($a_2/a_1 = 1.0$, $G/a_1 = 2.0$). The flow is in the direction of the temperature gradient $(\partial T/\partial Z)_\infty$. $-100\psi/(4\sqrt{2}K_1Bc^2) = \underline{0}$, 0.04, 0.15, 0.34, 0.61, 0.94, 1.33, 1.78, 2.29, 2.85, 3.46, 4.10 and $\underline{4.78}$. The underlined values are indicated in the figure.

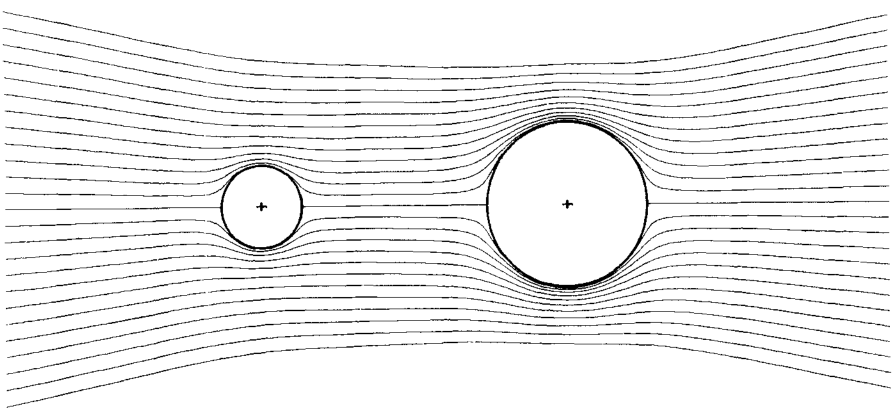

FIGURE 3. The streamlines for thermal-creep flow around two spheres ($a_2/a_1 = 0.5$, $G/a_1 = 2.3$). The flow is in the direction of the temperature gradient $(\partial T/\partial Z)_\infty$. $-100\psi/(4\sqrt{2}K_1Bc^2) = \underline{0}$, 0.03, 0.11, 0.24, 0.42, 0.65, 0.93, 1.25, 1.60, 2.00, 2.43, 2.90 and $\underline{3.39}$. The underlined values are indicated in the figure.

temperature T_0 is established. Consequently the gas velocity vanishes on the surface (at least to $O(k)$) because no other velocity slip occurs there owing to $\boldsymbol{u}_{\mathrm{H}}^{(0)} = 0$ (see (2.11)).

With the application of $\xi_2 \to 0$ in (4.10) with (4.11),

$$\left.\begin{aligned} A_n &= -2\sqrt{2}\,K_1Bc^2\frac{n(n+1)}{2n+1} \\ &\quad\times\frac{\sinh 2\xi_1 \sinh(n+\frac{1}{2})\,\xi_1-(2n+1)\sinh^2\xi_1\cosh(n+\frac{1}{2})\,\xi_1}{4\sinh^2(n+\frac{1}{2})\,\xi_1-(2n+1)^2\sinh^2\xi_1}\,\mathrm{e}^{-(n+\frac{1}{2})\,\xi_1}, \\ B_n &= -2\sqrt{2}\,K_1Bc^2\frac{n(n+1)}{2n+1}(2n+3)\frac{\sinh^2\xi_1\sinh(n+\frac{1}{2})\,\xi_1\,\mathrm{e}^{-(n+\frac{1}{2})\,\xi_1}}{4\sinh^2(n+\frac{1}{2})\,\xi_1-(2n+1)^2\sinh^2\xi_1}, \\ C_n &= -A_n, \quad D_n = -\frac{2n-1}{2n+3}B_n \end{aligned}\right\} \tag{4.16}$$

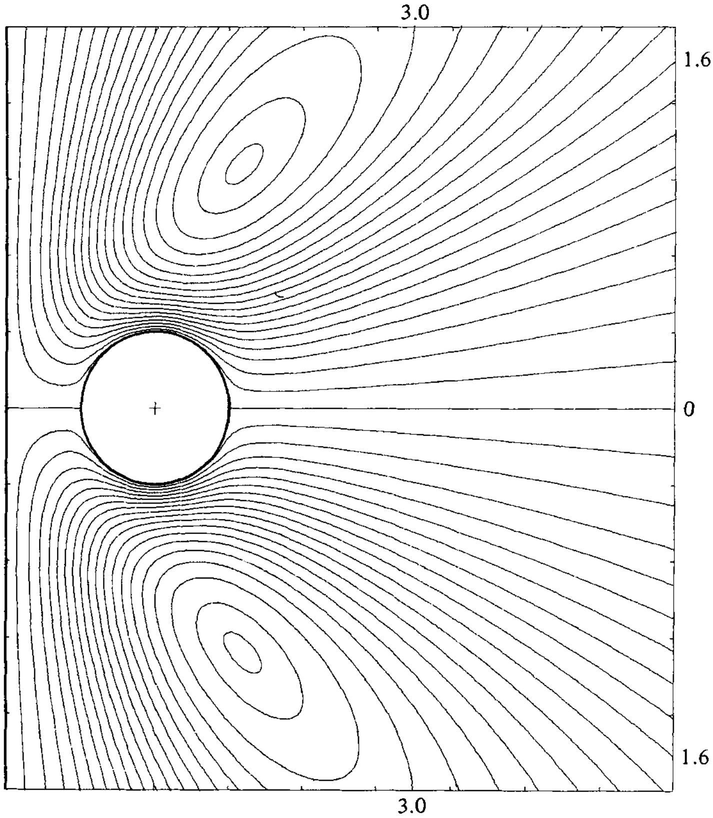

FIGURE 4. The streamlines for thermal-creep flow around a sphere in the presence of a plane wall ($G/a_1 = 1.0$). The flow is in the direction of the temperature gradient $(\partial T/\partial Z)_\infty$, i.e. from left to right if $(\partial T/\partial Z)_\infty > 0$, and vice versa if $(\partial T/\partial Z)_\infty < 0$. $-100\psi/(4\sqrt{2}K_1Bc^2) = \underline{0}$, 0.05, 0.2, 0.4, 0.6, 0.8, 1.0, 1.2, 1.4, $\underline{1.6}$, 1.8, 2.0, 2.2, 2.4, 2.6, 2.8, $\underline{3.0}$, 3.2, 3.4, and 3.5. The underlined values are indicated in the figure.

are obtained. Mention may be made here that these results hold whatever the thermal conductivity of the plane wall. A streamline pattern is shown in figure 4, where G is the distance between the plane and the nearest point of the sphere.

The total velocity field is obtained from (4.13)–(4.15) with (4.14*b*) and (4.15*b*) replaced respectively by $(h\,\partial z/\partial\alpha)_S = 0$ on the plane $(\xi_2 = 0)$ and $\eta = z/k$ near the plane.

4.2. *Eccentric-spheres case (S_2 encloses S_1)*

When S_2 is described by $\xi = \xi_2$ for $\xi_2 < \xi_1$, S_2 encloses S_1. The solutions in this case are obtained by simply replacing ξ_2 in all the expressions for b_4, Δ_n, A_{31}, A_{41}, …, A_{34} and A_{44} in (4.10) and (4.11) by $-\xi_2$, except for the exponential term in b_4. Figure 5 shows a typical stream line pattern for this case.

The total velocity is obtained by replacing (4.14*b*) with

$$\left(h\frac{\partial z}{\partial \alpha}\right)_S = -\frac{1}{\lambda}[\lambda^2-(z-z_2)^2]^{\frac{1}{2}} \quad \text{on} \quad S_2\,(\xi = \xi_2), \tag{4.17}$$

and (4.15*b*) with

$$\eta = \{\lambda-[r^2+(z-z_2)^2]^{\frac{1}{2}}\}/k \quad \text{near} \quad S_2, \tag{4.18}$$

other equations being left unchanged in (4.13)–(4.15).

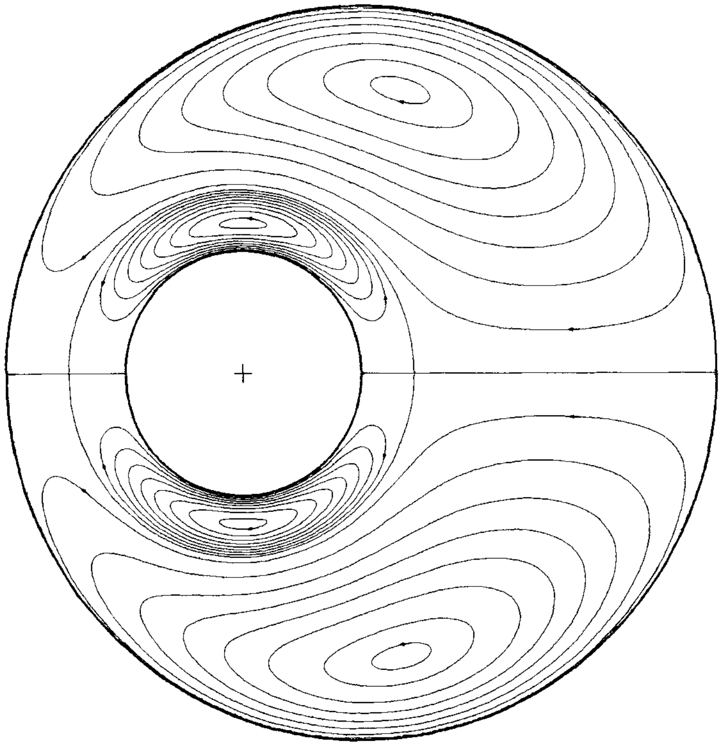

FIGURE 5. The streamlines for thermal-creep flow within the eccentric spheres ($a_2/a_1 = 3.0$, $G/a_1 = 1.0$). $(\partial T/\partial Z)_\infty$ is from left to right in this example, and the flow is in the direction indicated by arrows. The surfaces of the two spheres and a spherical surface within the flow domain indicated by a thin solid line are those for $\psi = 0$. Also two segments of the centreline are a line for $\psi = 0$. $100\psi/(4\sqrt{2}K_1 Bc^2) = -0.02, -0.04, -0.06, -0.08, -0.1, -0.12, -0.14$ and -0.16, as the size of the closed streamlines near the inner sphere becomes smaller. $100\psi/(4\sqrt{2}K_1 Bc^2) = 0.05, 0.2, 0.4, 0.6, 0.8, 1.0, 1.2$ and 1.3 as the size of the closed streamlines near the outer sphere becomes smaller.

5. Forces for two-spheres case

The force $\boldsymbol{F}$ acting on a sphere is

$$\boldsymbol{F} = \iint_A \boldsymbol{\sigma}\cdot\boldsymbol{n}\,\mathrm{d}A, \tag{5.1}$$

where A is a control surface that encloses the sphere. $\boldsymbol{\sigma}$ is the stress tensor (in the present case identical with minus the momentum-flux density tensor), and is given in the form (Sone 1971)

$$\boldsymbol{\sigma} = -P_0(\boldsymbol{I}+\boldsymbol{P}). \tag{5.2}$$

Here $\boldsymbol{I}$ is the unit tensor; $\boldsymbol{P}$ is the perturbation of the stress tensor from a uniform state and is split into two parts $\boldsymbol{P}_{\mathrm{H}}$ and $\boldsymbol{P}_{\mathrm{K}}$ as in (2.1). Since $\boldsymbol{P}_{\mathrm{K}}$ does not contribute to the total force (Sone & Tanaka 1980), the Hilbert part $\boldsymbol{P}_{\mathrm{H}}$ suffices for the description of $\boldsymbol{\sigma}$. Thus

$$\boldsymbol{\sigma} = P_0\{-(1+p_{\mathrm{H}})\boldsymbol{I} + k[\boldsymbol{\nabla u}_{\mathrm{H}} + (\boldsymbol{\nabla u}_{\mathrm{H}})^{\mathrm{T}}] - k^2\boldsymbol{\nabla\nabla}\tau_{\mathrm{H}} + \ldots\}. \tag{5.3}$$

p_{H}, $\boldsymbol{u}_{\mathrm{H}}$ and τ_{H} are to be expanded according to (2.2a), and the above expression is correct up to $O(k^2)$. Taking (4.1) into account, we have

$$\boldsymbol{\sigma} = -P_0\boldsymbol{I} + k^2P_0\{-p_{\mathrm{H}}^{(2)}\boldsymbol{I} + [\boldsymbol{\nabla u}_{\mathrm{H}}^{(1)} + (\boldsymbol{\nabla u}_{\mathrm{H}}^{(1)})^T]\} + O(k^3). \tag{5.4}$$

Now from the symmetry of the problem the forces acting on the spheres have only a z-component F. Following Stimson & Jeffery (1926) or Happel & Brenner (1965, chap. 4), we can obtain the expression for F in terms of ψ:

$$\frac{F}{P_0 a_1^2 k^2} = \pm\pi \int_{-1}^{1} \left\{ r^3 \frac{\partial}{\partial \xi}\left[\frac{1}{r^2}\mathscr{L}^2(\psi)\right]\right\}(1-\beta^2)^{\frac{1}{2}}\,\mathrm{d}\beta, \tag{5.5}$$

where a surface $\xi =$ constant is taken as A of (5.1), and the upper sign applies to $\xi =$ positive constant and the lower to $\xi =$ negative constant. With the use of (4.6) for ψ, the right-hand side of (5.5) reduces to

$$-\frac{2\sqrt{2}}{c}\pi \sum_{n=1}^{\infty} (2n+1)\,[A_n \pm B_n + C_n \pm D_n]. \tag{5.6}$$

It is convenient to introduce the non-dimensional force coefficients f_1 and f_2 defined by

$$\left.\begin{aligned} F_1 &= F_0 f_1, \quad F_2 = F_0 \lambda f_2, \\ F_0 &= 4\pi K_1 B P_0 a_1^2 k^2, \end{aligned}\right\} \tag{5.7}$$

where F_1 and F_2 are the forces acting on S_1 and S_2 respectively. F_0 denotes the force acting on the sphere with radius a_1 in a single-sphere problem when $\lambda_s = \lambda_g$. This can be expressed appropriately in terms of either the viscosity μ of the gas or its thermal conductivity λ_g,

$$F_0 = 8\pi K_1 C^* \left(\frac{\partial T}{\partial Z}\right)_\infty \left(\frac{8RT_0}{\pi}\right)^{-\frac{1}{2}} a_1^2 \mathbb{K}, \tag{5.8}$$

with $C^* = R\mu$ or $C^* = \frac{2}{5}\lambda_g$. Here the relation $\lambda_g = \frac{5}{2}R\mu$, which holds for the Boltzmann–Krook–Welander equation (see Vincenti & Kruger 1965, chap. X), has been used. The more general expression when $\lambda_s \neq \lambda_g$ for a single-sphere case has a factor $[1+(\lambda_g-\lambda_s)/(2\lambda_g+\lambda_s)]$ on both sides of (5.8) (Onishi 1972*b*).

Now using the expressions (4.10) for A_n, B_n, C_n and D_n together with (4.11), we obtain the expression for f_1 from (5.7):

$$\begin{aligned} f_1 = \sum_{n=1}^{\infty} 32n(n+1)\sinh\xi_1 \{&\sinh(n+\tfrac{1}{2})(\xi_1+\xi_2)\,[\sinh^2\xi_1\, \mathrm{e}^{(n+\frac{1}{2})\xi_2} - \sinh^2\xi_2\, \mathrm{e}^{-(n+\frac{1}{2})\xi_2}] \\ &-(2n+1)\sinh\xi_1 \sinh\xi_2 \sinh(\xi_1+\xi_2)\sinh(n+\tfrac{1}{2})\xi_1\}\frac{\mathrm{e}^{-(n+\frac{1}{2})\xi_1}}{\Delta_n}. \end{aligned} \tag{5.9}$$

The expression for f_2 can be derived from (5.9) by interchanging ξ_1 and ξ_2 with ξ_2 and ξ_1 respectively. When the two spheres are infinitely separated (i.e. ξ_1 and $\xi_2 \to \infty$)

$$f_1 \to 1 \quad \text{and} \quad f_2 \to 1, \tag{5.10}$$

as naturally expected. These coefficients f_1 and f_2 are functions of ξ_1 and ξ_2 only, and therefore functions of λ and g, g being the non-dimensional separation between the spheres defined by $g = G/a_1$. ξ_1 and ξ_2 are related to λ and g as

$$\left.\begin{aligned} \xi_1 &= \cosh^{-1} z_1 = \ln[z_1 + (z_1^2-1)^{\frac{1}{2}}], \quad \xi_2 = \cosh^{-1}\frac{z_2}{\lambda}, \\ z_1 &= \frac{(1+\lambda+g)^2+1-\lambda^2}{2(1+\lambda+g)}, \quad z_2 = (\lambda^2-1+z_1^2)^{\frac{1}{2}}. \end{aligned}\right\} \tag{5.11}$$

The numerical values of f_1 and f_2 for $\lambda \leqq 1$ have been calculated for various values of g.† Those for $\lambda > 1$ can be obtained from the relation

$$f_2(g,\lambda) = f_1(g\lambda^{-1}, \lambda^{-1}), \quad \text{or} \quad f_1(g,\lambda) = f_2(g\lambda^{-1}, \lambda^{-1}), \tag{5.12}$$

† Copies of the tables may be obtained on request from the author.

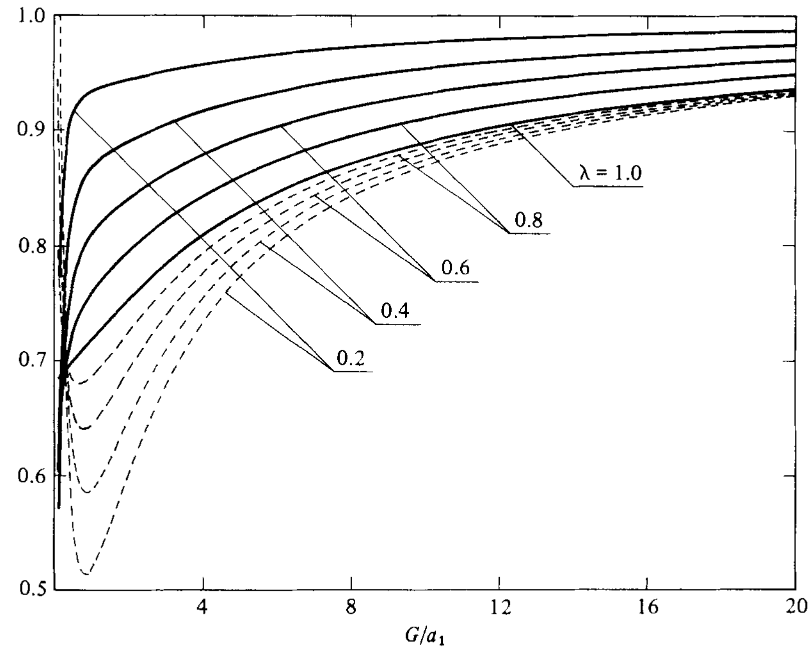

FIGURE 6. The force coefficient on each sphere for various values of the radius ratio $\lambda = a_2/a_1$ (two-sphere problem for thermal creep). Thick solid lines: f_1. Thin dashed lines: f_2.

which can easily be deduced from physical considerations. The graphs for f_1 and f_2 versus g are shown in figure 6, where we notice that the forces acting on the smaller spheres have the minimum values at certain separations ($g \approx 0.6$–1.0) which depend on λ. In contrast with this, for two unequal spheres moving along the line of centres the forces become minimum when they act on the larger spheres according to the calculation by Cooley & O'Neill (1969*a*) based on Stimson & Jeffery's (1926) solution.

The convergence of the series (5.9) is good when the two spheres are far apart, but it becomes very slow as they become close together, owing to the term $\exp(\pm n\xi_{1,2})$. When the two spheres are touching, we may convert the above series into an integral form by taking appropriate limit, i.e. letting $\xi_1, \xi_2 \to 0$, while retaining $n\xi_1$, $n\xi_2$ and $\sinh\xi_1/\sinh\xi_2$ as $O(1)$. The result is

$$\lim_{\substack{\xi_1\to 0\\ \xi_2\to 0}} f_1 = \int_0^\infty \frac{2x^2\sinh\nu x[\lambda^2 e^{(\nu-2)x} - e^{-\nu x}] - 2(1+\lambda)\,x^3[1-e^{-2x}]}{\lambda^2[\sinh^2\nu x - (\nu x)^2]}\,dx, \tag{5.13}$$

Where $\nu = (1+\lambda)/\lambda$. A similar integral is also obtained for $\lim f_2$. It is clear that the integral in (5.13) diverges logarithmically unless $\lambda = 1$ (or $\nu = 2$), because the integrand behaves like $6\lambda(\lambda-1)(\lambda+1)^{-2}/x$ for small x. It should be noted, however, that the sum of the two forces acting on S_1 and S_2, i.e. F_1+F_2, always remains finite, for the singular behaviour of $\lim f_1$ is cancelled out by the corresponding behaviour of $\lim\lambda f_2$. For a special case $\lambda = 1$ the integral (5.13) becomes simple, converging to yield

$$\lim_{\xi_1\to 0} f_1 = \frac{1}{4}\int_0^\infty \frac{x^2(1-e^{-x})}{\sinh x + x}\,dx = 0.679920. \tag{5.14}$$

Attention must be drawn to the following point: when the two spheres become close together the mean free path of gas molecules becomes comparable to, or even larger than, the gap between the spheres, which leads to violation of assumption (iii) made in §1. Therefore the present analysis does not hold for the case in which the two spheres are close together. Naturally, the expressions (5.13) and (5.14) do not make a physically meaningful point but merely a mathematical one. In spite of this, however, it would be of some significance to examine the behaviour of the series (5.9) as ξ_1 and $\xi_2 \to 0$.

When the spheres are not constrained they move in the direction opposite to the temperature gradient imposed at infinity. We wish to know what velocities the spheres will have under the thermal force (5.7).† Suppose that the spheres S_1 and S_2 were translating in the gas along the line of centres under the external forces F_{e1} and F_{e2} respectively. Since the gas motion caused by these spheres is governed by equations of Stokes type, we may express these forces in the form (see e.g. Brenner & O'Neill 1972; Jeffrey & Onishi 1984),

$$\left.\begin{aligned} F_{e1} &= 6\pi\mu a_1[X_{11}(g,\lambda)\,W_1 + \tfrac{1}{2}(1+\lambda)\,X_{12}(g,\lambda)\,W_2],\\ F_{e2} &= 6\pi\mu a_2\left[\frac{1+\lambda}{2\lambda}X_{21}(g,\lambda)\,W_1 + X_{22}(g,\lambda)\,W_2\right], \end{aligned}\right\} \tag{5.15}$$

where W_1 and W_2 are respectively the velocities of S_1 and S_2 in the direction of increasing z. Inverting the above expressions, we may also write for W_1 and W_2 (Batchelor 1976, 1982; Jeffrey & Onishi 1984)

$$\left.\begin{aligned} W_1 &= \frac{1}{6\pi\mu a_1}\left[x_{11}(g,\lambda)\,F_{e1} + \frac{2}{1+\lambda}x_{12}(g,\lambda)\,F_{e2}\right],\\ W_2 &= \frac{1}{6\pi\mu a_2}\left[\frac{2\lambda}{1+\lambda}x_{21}(g,\lambda)\,F_{e1} + x_{22}(g,\lambda)\,F_{e2}\right]. \end{aligned}\right\} \tag{5.16}$$

The sets of non-dimensional scalar quantities $(X_{11}, X_{12}, X_{21}, X_{22})$ and $(x_{11}, x_{12}, x_{21}, x_{22})$ are respectively, those commonly called the resistance functions and mobility functions for axisymmetric translational motions of two spheres, and it may be noted that the following relations hold:

$$\left.\begin{aligned} &X_{22}(g,\lambda) = X_{11}(g\lambda^{-1},\lambda^{-1}),\\ &X_{21}(g,\lambda) = X_{12}(g,\lambda) = X_{21}(g\lambda^{-1},\lambda^{-1}) = X_{12}(g\lambda^{-1},\lambda^{-1}),\\ &x_{22}(g,\lambda) = x_{11}(g\lambda^{-1},\lambda^{-1}),\\ &x_{21}(g,\lambda) = x_{12}(g,\lambda) = x_{21}(g\lambda^{-1},\lambda^{-1}) = x_{12}(g\lambda^{-1},\lambda^{-1}). \end{aligned}\right\} \tag{5.17}$$

A comprehensive account of these scalar functions is given by Jeffrey & Onishi (1984), and some of the numerical values of x_{ij} $(i,j = 1 \text{ or } 2)$ are listed in that paper as $x_{ij}^a(s,\lambda)$, where $s = 2(1+\lambda+g)/(1+\lambda)$.

Substitution of $F_{e1} = F_1$ and $F_{e2} = F_2$ into (5.16) yields

$$W_1 = \frac{F_0}{6\pi\mu a_1}x_1^{\mathrm{T}}(g,\lambda), \quad W_2 = \frac{F_0}{6\pi\mu a_1}x_2^{\mathrm{T}}(g,\lambda), \tag{5.18}$$

where

$$x_1^{\mathrm{T}} = x_{11}f_1 + \frac{2\lambda}{1+\lambda}x_{12}f_2, \quad x_2^{\mathrm{T}} = \frac{2}{1+\lambda}x_{21}f_1 + x_{22}f_2, \tag{5.19}$$

and $$x_1^{\mathrm{T}}, x_2^{\mathrm{T}} \to 1 \quad \text{as} \quad g \to \infty.$$

† The effect of unsteadiness due to the sphere motions through the gas with the uniform temperature gradient is found to be of order B^2, which is totally negligible in the present linearized theory (see e.g. Sone & Aoki 1977).

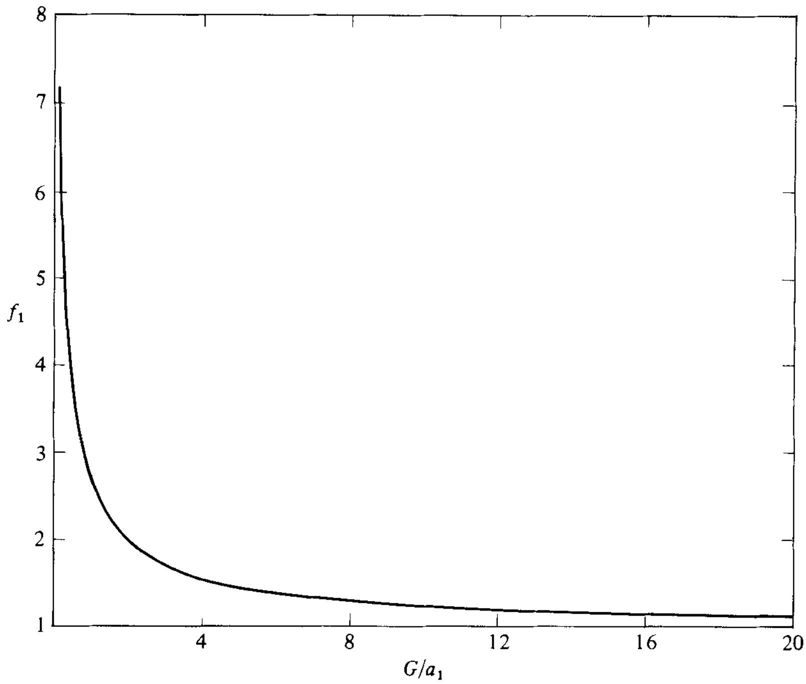

FIGURE 7. The force coefficient on the sphere (sphere–plane problem for thermal creep).

Note that the relation $x_2^{\mathrm{T}}(g,\lambda) = x_1^{\mathrm{T}}(g\lambda^{-1},\lambda^{-1})$ holds. Some values of $x_1^{\mathrm{T}}(g,\lambda)$ and $x_2^{\mathrm{T}}(g,\lambda)$ are as follows: for $\lambda = 1.0$, $x_1^{\mathrm{T}} = x_2^{\mathrm{T}} = 1.02768$ at $g = 1.0$, $x_1^{\mathrm{T}} = x_2^{\mathrm{T}} = 1.00752$ at $g = 3.0$; for $\lambda = 0.5$, $x_1^{\mathrm{T}} = 1.00470$, $x_2^{\mathrm{T}} = 1.06007$ at $g = 1.0$, and $x_1^{\mathrm{T}} = 1.00126$, $x_2^{\mathrm{T}} = 1.01086$ at $g = 3.0$ (more data may be obtained from the author on request). Both x_1^{T} and x_2^{T} are larger than unity. The above expressions give the translational velocities that would be acquired by S_1 and S_2 when they are free to move under the action of the given thermal force (5.7). It is seen that the smaller sphere tends to move faster than the larger one, because x_1^{T} is smaller than x_2^{T} for $\lambda < 1$. We mention incidentally that $-F_0/6\pi\mu a_1$ is approximately 6.1 $\mathbb{K}$ (mm/s) for air at $T_0 = 0$ °C when the temperature gradient is one degree over $100a_1$, i.e. $a_1(\partial T/\partial Z)_\infty = 0.01$ °C.

6. Forces for the other cases

We now turn our attention to the forces for the other two cases, which are obtainable from the results of §5.

6.1. *Sphere–plane case* (*when S_2 reduces to a plane wall*)

With the application of $\xi_2 \to 0$ in (5.9), f_1 becomes

$$f_1 = \sum_{n=1}^{\infty} 8n(n+1)\frac{\sinh^3\xi_1 \sinh(n+\frac{1}{2})\xi_1\, e^{-(n+\frac{1}{2})\xi_1}}{4\sinh^2(n+\frac{1}{2})\xi_1 - (2n+1)^2\sinh^2\xi_1}, \tag{6.1}$$

and also

$$f_1 \to 1 \quad \text{as} \quad \xi_1 \to \infty. \tag{6.2}$$

The graph for f_1 is shown in figure 7. The total force F_2 acting on the plane wall is equal to but opposite in sign to the force F_1 acting on the sphere, as can be checked from (4.16) and (5.6). When the sphere is close to the plane (i.e. $\xi_1 \to 0$), the above series diverges logarithmically, because the integrand of (5.13), when the limit process $\lambda \to \infty$ (or $\nu \to 1$) is applied, behaves like $1/x$ for small x.

When the sphere is left unconstrained in the gas, it will acquire the velocity due to the thermal force F_1. The relation between the velocity of the sphere and the external force on it, on the other hand, is obtained by formally putting $W_2 = 0$ and $\lambda \to \infty$ in (5.15) as

$$F_{e1} = 6\pi\mu a_1 X_{11}(g)\, W_1, \tag{6.3}$$

where $g = G/a_1$ (G is the separation between the plane wall and the nearest point of the sphere). Brenner (1961) has calculated the force coefficient on a sphere moving toward a plane wall, and it is clearly seen that the present scalar function $X_{11}(g)$ is none other than this force coefficient. Thus $X_{11}(g)$ is given from Brenner as (see λ in his notation)

$$X_{11}(g) = \tfrac{4}{3}\sinh\xi_1 \sum_{n=1}^{\infty} \frac{n(n+1)}{(2n-1)(2n+3)}\left\{\frac{2\sinh(2n+1)\xi_1 + (2n+1)\sinh 2\xi_1}{4\sinh^2(n+\frac{1}{2})\xi_1 - (2n+1)^2\sinh^2\xi_1} - 1\right\}, \tag{6.4}$$

Equating the external force F_{e1} to the thermal force F_1 given in (5.7) together with (6.1) for f_1, we have from (6.3)

$$W_1 = \frac{F_0}{6\pi\mu a_1} x_1^{\mathrm{T}}(g), \quad x_1^{\mathrm{T}} = \frac{f_1}{X_{11}}, \tag{6.5}$$

and

$$x_1^{\mathrm{T}} \to 1 \quad \text{as} \quad g \to \infty.$$

The values of f_1, X_{11} and x_1^{T} are also available (from the author on request). However, to get a rough idea for the velocity of the sphere under the thermal force, we give some examples of the values of $x_1^{\mathrm{T}}(g)$: $x_1^{\mathrm{T}}(0.2) = 0.64578$, $x_1^{\mathrm{T}}(0.6) = 0.87385$, $x_1^{\mathrm{T}}(1.0) = 0.93788$, $x_1^{\mathrm{T}}(2.0) = 0.98183$, $x_1^{\mathrm{T}}(5.0) = 0.99771$.

6.2. *Eccentric-spheres case (when S_2 encloses S_1)*

In this case, we have

$$f_1 = \sum_{n=1}^{\infty} 16n(n+1)\sinh\xi_1[\sinh^2\xi_1 - \sinh^2\xi_2 + (2n+1)\sinh\xi_1\sinh\xi_2\sinh(\xi_1-\xi_2)]$$
$$\times [\mathrm{e}^{-(2n+1)\xi_2} - \mathrm{e}^{-(2n+1)\xi_1}]/\Delta_n^*(\xi_1,\xi_2), \tag{6.6}$$

where

$$\Delta_n^*(\xi_1,\xi_2) = \Delta_n(\xi_1,-\xi_2) = 16\sinh^2(n+\tfrac{1}{2})(\xi_1-\xi_2) - 4(2n+1)^2\sinh^2(\xi_1-\xi_2), \tag{6.7}$$

and

$$f_1 \to 1 \quad \text{as} \quad \xi_1 \to \infty \quad (\text{or } \lambda \to \infty). \tag{6.8}$$

The graphs of f_1 for some values of λ are plotted in figure 8.

The total force acting on S_2 by the gas is equal in magnitude but opposite in sign to that acting on S_1, i.e. $F_2 = -F_1$ or $f_2 = -f_1/\lambda$. This is easily checked by considering the integral of the stress divergence over a volume V bounded by two surfaces A_1 and A_2, both of which lie completely within the gas and enclose S_1, and converting the integral by Gauss' theorem to surface integrals on A_1 and A_2, each of which eventually approaches S_1 and S_2 respectively. For ξ_1 and $\xi_2 \to 0$ an integral expression similar to (5.13) can also be obtained for the above series (6.6), and diverges logarithmically, since the integrand behaves like $6\lambda(\lambda+1)(\lambda-1)^{-2}/x$ for small x.

The sphere S_1, if it is free within the stationary enclosing sphere S_2, will move under the thermal force. Cooley & O'Neill (1969*b*) have given the force coefficients for the eccentric-spheres case when they are moving toward each other through an inertialess

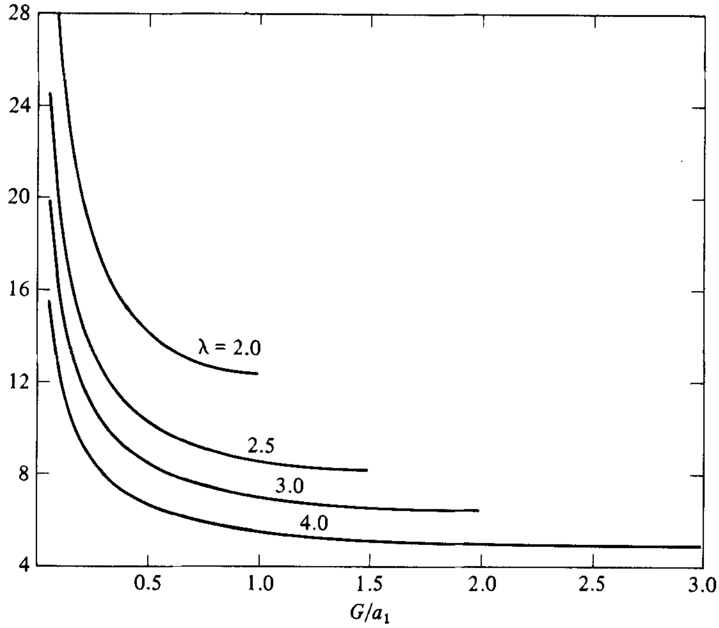

FIGURE 8. The force coefficient on the inner sphere for various values of the radius ratio $\lambda = a_2/a_1$ (eccentric-spheres problem for thermal creep).

continuum fluid with the same speed. Since rigid-body motion of the gas is achieved when the eccentric spheres are moving with the same speed in the same direction, we may calculate X_{11}, X_{12}, X_{21} and X_{22} in (5.15) from the results of Cooley & O'Neill (1969*b*) and from the rigid-body motion:

$$\left.\begin{aligned} X_{11}(g,\lambda) &= \lambda X_{22}(g,\lambda) = \tfrac{1}{2}\mathscr{f}(g,\lambda), \\ X_{12}(g,\lambda) &= X_{21}(g,\lambda) = -\frac{2}{1+\lambda}X_{11}(g,\lambda), \end{aligned}\right\} \tag{6.9}$$

where $\mathscr{f}(g,\lambda)$ is the force coefficient on the inner sphere for eccentric spheres approaching each other, and is given in table 3 of Cooley & O'Neill (1969*b*) by the notation F_1. Unfortunately, Cooley & O'Neill have not given the expression for $\mathscr{f}$ (or F_1 in their notation) but only a small number of the numerical values, so we think it worthwhile to give the explicit expression for it for further possible use:

$$\left.\begin{aligned} X_{11}(g,\lambda) = \tfrac{16}{3}\sinh\xi_1 \sum_{n=1}^{\infty} & \frac{n(n+1)}{(2n-1)(2n+3)}\{(2n-1)(2n+3) \\ & \times[\sinh^2\xi_1\,\mathrm{e}^{-(2n+1)\xi_2}-\sinh^2\xi_2\,\mathrm{e}^{-(2n+1)\xi_1}] \\ & +[(2n+1)\sinh 2\xi_1+2\cosh 2\xi_1]\,\mathrm{e}^{-(2n+1)\xi_2} \\ & -[(2n+1)\sinh 2\xi_2+2\cosh 2\xi_2]\,\mathrm{e}^{-(2n+1)\xi_1}\}/\Delta_n^*(\xi_1,\xi_2), \end{aligned}\right\} \tag{6.10}$$

where $\Delta_n^*(\xi_1,\xi_2)$ is given in (6.7), and the relation between (g,λ) and (ξ_1,ξ_2) is given by (5.11), except that z_1 there should be replaced by $z_1 = [(1-\lambda+g)^2+1-\lambda^2]/[2(1-\lambda+g)]$. The sphere S_1 will then move with the velocity

$$W_1 = \frac{F_0}{6\pi\mu a_1}x_1^{\mathrm{T}}(g,\lambda), \quad x_1^{\mathrm{T}} = \frac{f_1}{X_{11}}, \tag{6.11}$$

under the thermal force, which is given by (5.7) together with (6.6) for f_1. The numerical values of f_1, X_{11} and x_1^{T} are available (from the author on request). The following numerical examples of $x_1^{\mathrm{T}}(g, \lambda)$, say, for $\lambda = 3.0$ will give some idea of the velocity W_1: $x_1^{\mathrm{T}} = 0.98283$ at $g = 0.2$, 1.48773 at $g = 0.5$, 1.89481 at $g = 1.0$, 2.08894 at $g = 1.5$, and 2.14874 at $g = 1.99$. Incidentally, it should be pointed out that the results for the eccentric-spheres case reproduce the corresponding ones for the sphere-plane case.

Finally, it should be noted that assumption (ii) of §1 can be extended to a more general case in which some of the molecules impinging on the surfaces are specularly reflected, i.e. boundary conditions of specular–diffusive type or Maxwell type hold. The simple replacement of the present set (K_1, $Y_1(\eta)$) with the corresponding one given by Onishi (1972*a*) i.e. $(-\frac{1}{2}d, -Q(\eta))$ or its refined version $(-\frac{1}{2}\kappa_{\mathrm{T}}, -Q_{\mathrm{T}}(y))$ (Onishi 1973), leads to the desired results for this case.

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